INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids, **26**: 877–885 (1998).

A MIXED GALERKIN METHOD FOR COMPUTING THE FLOW BETWEEN ECCENTRIC ROTATING CYLINDERS

E. KIM
School of Mechanical Engineering, Pusan National University, Pusan 609-735, South Korea

SUMMARY

A mixed Galerkin technique with B-spline basis functions is presented to compute two-dimensional incompressible flow in terms of the primitive variable formulation. To circumvent the Babuska–Brezzi stability criterion, the artificial compressibility formulation of the equation of mass conservation is employed. As a result, the diagonal components of the matrix form in the governing equations are not singular. The B-spline basis is used because it is superior to other splines in providing computer solutions to fluid flow problems. One of the advantages of the B-spline basis is that it has excellent approximation properties. Numerical examples of applications of the mixed formulation are presented to demonstrate the convergence characteristics and accuracy of the present formulation. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: B-spline basis; Galerkin method; Gauss–Newton method; mixed formulation

1. INTRODUCTION

A mixed form is recognized as a successful formulation for interpolation of velocity and pressure. Even though its application in fluid problems is rather recent, it shows many advantages over the finite difference method. Babuska[1] provided a mathematical framework for the behaviour of a mixed method, but this formulation gave poor accuracy when applied to a second-order elliptic equation with large coefficients of the first-order terms. To overcome these kinds of difficulties, Abrahamsson *et al.*[2] gave a theoretical treatment. Based on Abrahamsson's theory, a one-dimensional problem was solved by Christie *et al.*[3] this was extended to two dimensions by Heinrich *et al.*[4] Keller *et al.*[5] significantly advanced the development of the upwind Petrov–Galerkin method, which considerably reduces the amount of numerical diffusion. Hughes *et al.*[6] published a new formulation with better stability properties than the classical Galerkin method. They applied an ingenious weighting procedure that results in a stable system of the Petrov–Galerkin formulation. Recently, Sampaio[7] developed the least squares Galerkin formulation for the transient equations. Zienkiewicz and Wu[8] noted that the general stabilized forms lead to non-symmetric steady state equations.

In this paper we shall solve the steady state exact governing equations in mixed formulation through the use of the Galerkin method. However, direct discretization of the steady state Navier–Stokes problem would lead to a singular system. To circumvent this problem, we work with the artificial compressibility formulation of the conservation of mass, adjoining its time-asymptotic form to the steady state Navier–Stokes equations. The artificial compressibility formulation is used with

Correspondence to: E. Kim, School of Mechanical Engineering, Pusan National University, Pusan 609-735, South Korea.

CCC 0271–2091/98/080877–09 $17.50

Received January 1996
Revised January 1997

the B-spline basis. Panton and Sallee[9] gave a good comparison among three types of splines. The resulting algebraic equations are solved by employing the Gauss–Newton method.[10]

2. MATHEMATICAL ANALYSIS

The physical situation considered is that of steady laminar flow between two cylinders. The thermodynamical properties of the fluid are assumed constant. The non-dimensional governing equations of linear momentum and mass can be written as

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2}$$

where u_i is the two-dimensional velocity field, τ_{ij} is the viscous stress tensor and p is the pressure.

In the classical mixed formulation the structure of the discretized equation is

$$\begin{bmatrix} \mathrm{K} & \mathrm{Q} \\ \mathrm{Q}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathrm{u} \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{F} \\ \mathbf{0} \end{bmatrix}. \tag{3}$$

In equation (3) the system is singular owing to the zero diagonal component, which entails special treatment of the system. To avoid the difficulties posed by the classical form, the artificial compressibility formulation of the equation of continuity is used. Although our professed interest lies in the steady state problem, we write the conservation equations for momentum and mass in their unsteady (non-dimensional) forms as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} = 0, \tag{4}$$

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{\partial u_i}{\partial x_i} = 0, \tag{5}$$

where c is the speed of sound.

For the conservation of mass, we time-discretize equation (5) as

$$\frac{1}{c^2}\frac{p^{(n+1)} - p^{(n)}}{\Delta t} = -\frac{\partial u_j^{(n+1/2)}}{\partial x_j}. \tag{6}$$

At the $n + \frac{1}{2}$ time level the equation of momentum becomes

$$u_i^{(n+1/2)} = u_i^{(n)} + \frac{\Delta t}{2}\left(-u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_i}\right)\Bigg|_n. \tag{7}$$

On substituting equation (7) into equation (6), we obtain

$$\frac{1}{c^2}(p^{(n+1)} - p^{(n)}) = -\Delta t\left[\frac{\partial u_i}{\partial x_j} + \frac{\Delta t}{2}\left(-u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_i}\right)\right]\Bigg|_n. \tag{8}$$

In equations (7) and (8) the steady state is characterized by $\partial u/\partial t = 0$ and $\partial v/\partial t = 0$ in equations (4) and (5) and by $p^{(n+1)} = p^{(n)}$ in equation (8). By the standard Galerkin procedure we solve the following system of equations:

$$u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_i} = 0, \tag{9}$$

$$\frac{\partial u_i}{\partial x_j} + \delta\left(-u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_i}\right) = 0. \tag{10}$$

Here δ is a suitably small number. Note that in the limit $\delta \to 0$ we recover the steady Navier–Stokes problem. The dependence of the solution on the value of δ, typically of $O(10^{-9})$ and leading to RMS error in mass conservation of $O(10^{-8})$, is discussed by Zienkiewicz and Wu[8] for theoretical convergence studies.

We introduce interpolation functions N_{u_i} and N_p such that

$$u_i \approx N_{u_i,i} u_i, \tag{11}$$

$$p \approx N_{p,i}\, p, \tag{12}$$

where N_{u_i} and N_p are the velocity and pressure interpolation functions respectively.

In equations (11) and (12) the B-spline basis is used as a weight factor. We partition the interval $[0, 1]$ as

$$\pi : 0 = z_1 < z_2 < \cdots < z_l < z_{l+1} = 1,$$

where z_i are the breakpoints. $P_{k,\pi}$ is the linear space and can be expressed as

$$P_{k,\pi} = \{f(z) : f(z) = p_i(z) \text{ if } z \in [z_i, z_{i+1}],\ 1 \leqslant i \leqslant l\}. \tag{13}$$

For each subinterval $[z_i, z_{i+1}]$, $P_{k,\pi}$ has a maximum order of k. Since there are l subintervals, the dimension of $P_{k,\pi}$ is kl. At each of the interval midpoints z_i, $2 \leqslant i \leqslant l$, we introduce $S_{k,\pi,v}$, a subspace of $p_{k,\pi}$. Here $v = \{v_i\}_{i=2}^{l}$ is the smoothing index of the subspace $S_{k,\pi,v}$. Now we construct the basis of $S_{k,\pi,v}$ which has local support. To obtain this kind of basis, we generate the recurrence relation of the B-spline basis as[11]

$$\begin{aligned} B_{i,k}(z) &= \frac{z - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(z) + \frac{t_{i+k} - z}{t_{i+k} - t_{i+k+1}} B_{i+1,k-1}(z), \\ B_{j,1} &= \begin{cases} 1 & \text{for } z \in [t_j, t_{j+1}], \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{14}$$

In equation (14), $t = \{t\}_1^{n+k}$ is a non-decreasing sequence as follows:

(i) $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k \leqslant x_1$ and $x_{i+1} \leqslant t_{l+1} \leqslant \cdots \leqslant t_{n+k}$
(ii) z_i occurs exactly $k - v_i$ times in t, $2 \leqslant i \leqslant l$.

The B-spline indicated above have the properties

$$\begin{aligned} &B_i(z) \geqslant 0, \quad 1 \leqslant i \leqslant N, \quad && z \in [z_1, z_{l+1}], \\ &\sum_{i=1}^{N} B_i(z) = 1, && z \in [z_1, z_{l+1}], \end{aligned} \tag{15}$$

and the relevant relations

$$\begin{array}{lll} B_1(z_1) = B_n(z_{l+1}) = 1, & B_j(z_1) = 0, \quad j > 1, & B_j(z_{l+1}) = 0, \quad j < n, \\ B_j(z) = 0, \quad z \notin [t_j, t_{j+k}], & B_j(z) \geqslant 0, \quad z \in [0, 1], & B_1'(z_i) + B_2'(z_i) = 0, \\ B_j'(z_1) = 0, \quad j > 2, & B_{n-1}'(z_{l+1}) + B_n'(z_{l+1}) = 0, & B_j'(z_{l+1}) = 0, \quad j < n-1. \end{array} \tag{16}$$

In the present calculations we employ a cubic B-spline basis, i.e. order $k = 4$, and a knot sequence

$$z_1 = t_1 = t_2 = t_3 = t_4\,, z_2 = t_5, \quad \cdots z_l = t_n, z_{l+1} = t_{n+1} = \cdots = t_{n+4}.$$

We introduce two independent basis functions to satisfy the solution of the flow between two cylinders $\{A_i(x)\colon 1 \leqslant i \leqslant N_x\}$ is the regular set of B-splines in the x-direction and $\{b_j(y) : 1 \leqslant j \leqslant N_y\}$ is the periodic B-spline basis in the y-direction. The interpretation of the regular B-spline is mentioned above and the requirement for the solution of the periodic B-spline is[11]

$$\Phi = \begin{bmatrix} I_3 & 0 & \Psi \\ 0 & I_{N_y-6} & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} c & 2 & 1 \\ -c & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{17}$$

where $c = 2B_2''n(0)/B_3''(0)$ and I_3 and I_{N_y-6} are unit matrices. The new base vectors $\{b_i\}_{i=1}^{N_y-3}$ are given by

$$b = \Phi B = (b_1, b_2, \ldots, b_{N_y-3})^{\mathrm{T}}, \tag{18}$$

where $B = (B_1, B_2, \ldots, B_{N_y-3})^{\mathrm{T}}$.

Figure 1 shows the regular B-spline basis functions and Figure 2 shows the periodic B-spline basis functions.

The non-linear algebraic system representing the governing equations (9) and (10) can be expressed as

$$G(\mathrm{u}, \lambda) = 0. \tag{19}$$

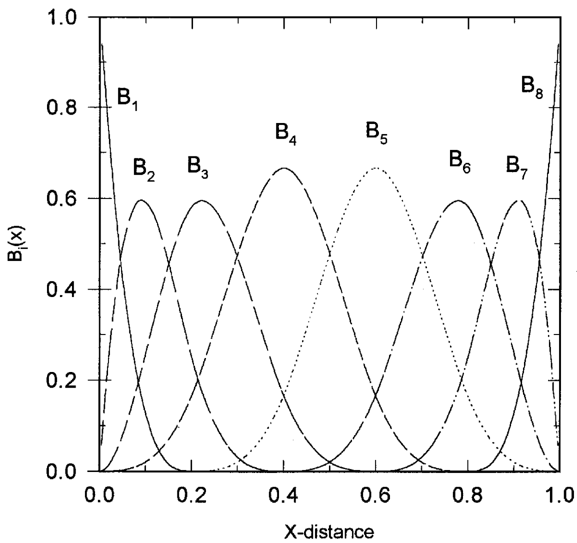

Figure 1. Regular B-spline basis functions, $k = 3, N = 5$

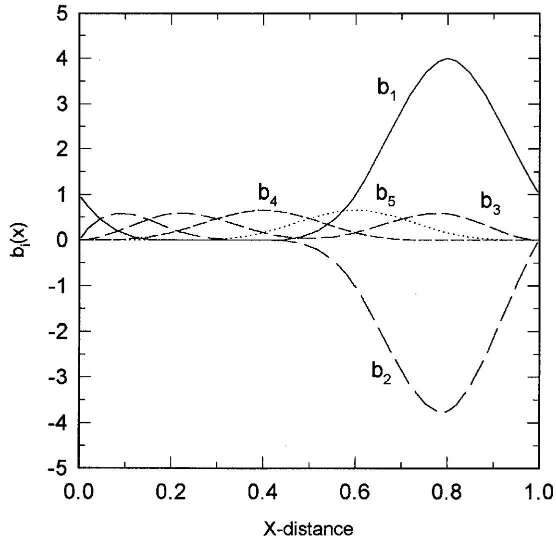

Figure 2. Periodic B-spline basis functions, $k = 3, N = 5$

Here G: $U \oplus \Lambda \subset \mathbb{R}^n \to \mathbb{R}^m$, $\dim U = m$, $\dim \Lambda = n - m$; $u \in U$ is the vector of unknown variables and $\lambda \in \Lambda$ denotes the parameters. To get a solution, we vary one parameter while fixing the other parameters. Therefore $n - m = 1$ and the regular manifold of equation (19) is the path. For the iterative calculation of equation (19) the Gauss–Newton method is employed.

3. NUMERICAL RESULTS

In this section, two numerical examples are presented to demonstrate the convergence and accuracy characteristics of the mixed Galerkin formulation with B-spline basis functions. For each of the two examples, comparisons with analytical and numerical results available in the literature are given. Before source code compilation, Galerkin coefficients are first implemented. Howle[12] mentioned two areas of savings for this procedure. First, the number of floating point operations per iteration is significantly reduced while the number of iterations required for convergence is unchanged. Second, it is no longer required to store the inner product coefficient. All numerical computations were performed on VAX8650 using double-precision accuracy. The calculations required 3–30 min CPU time for spline numbers $N = 12$, 14 and 16 and took three to seven iterations to converge.

3.1. Viscous flow between concentric rotating cylinders

To test the correctness of the solution, it was compared with the analytical solution for a simple case. In this case the outer cylinder is stationary while the inner cylinder rotates with angular velocity ω. The flow is one-dimensional and the results can be compared with the analytical solution for the tangential velocity and pressure field. For values of the Reynolds number from $Re = 1$ to 1000, Krakow[13] presented this problem. He obtained impressive accuracy and convergence compared with previous papers. His results, however, cannot be compared with analytical solutions in this analysis, because he eliminated the pressure term by mathematical manipulation. Here, for comparison,[14] the

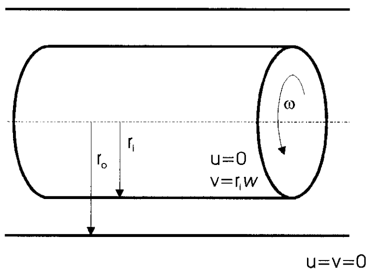

Figure 3. Schematic diagram of concentric rotating cylinders

full Navier–Stokes equations in primitive variable formulation are used. After applying the boundary conditions in Figure 3, we have B-spline expansions

$$u(x, y) = \sum_{i=2}^{N_x} \sum_{j=1}^{N_y-3} u_{ij} A_i(x) b_j(y), \tag{20}$$

$$v(x, y) = v_1 A_1(x) + \sum_{i=2}^{N_x} \sum_{j=1}^{N_y-3} v_{ij} A_i(x) b_j(y), \tag{21}$$

$$p(x, y) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y-3} p_{ij} A_i(x) b_j(y). \tag{22}$$

We substitute expansions (20)–(22) into the discretized governing equations to obtain solutions using the Galerkin method with the spline basis $A \otimes b$. Although the analytical solution is independent of the Reynolds number, the concentric rotating flow is calculated at $Re = 50$. Figure 4

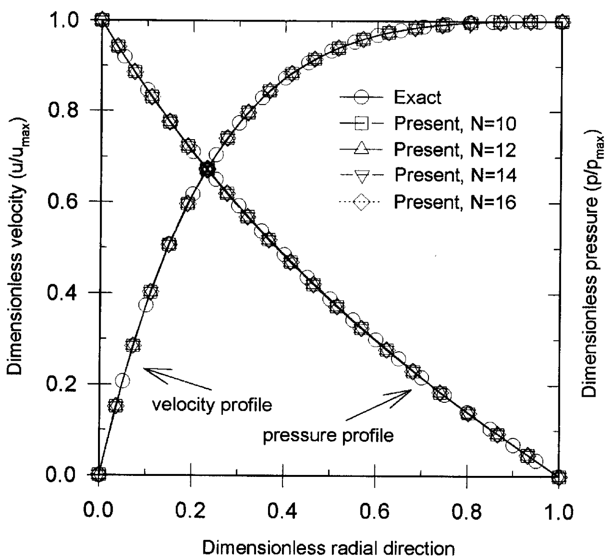

Figure 4. Velocity and pressure distribution along radial direction for $\theta = 87{\cdot}5°$ and $Re = 50$

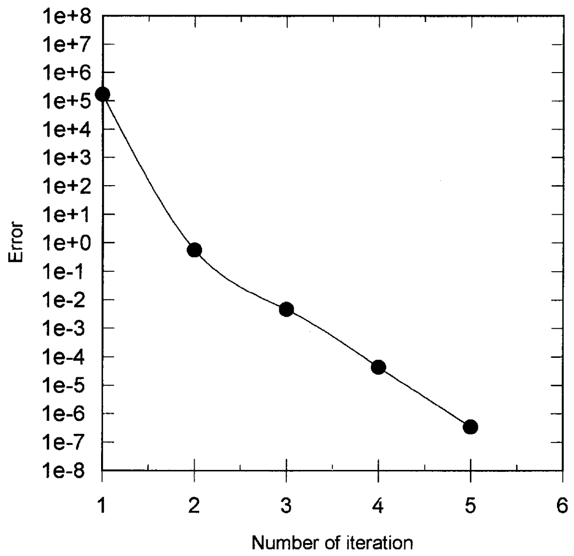

Figure 5. Convergence of solution

shows the velocity and pressure distribution along the diagonal direction for $\theta = 87{\cdot}5°$. We performed the calculation with different numbers of B-splines. The results show very good agreement with the analytical solution. The plots of the two results in Figure 4 are hardly distinguishable. Figure 5 shows the convergence of the solution procedure. It is worth noting that convergence has been reached within five iterations.

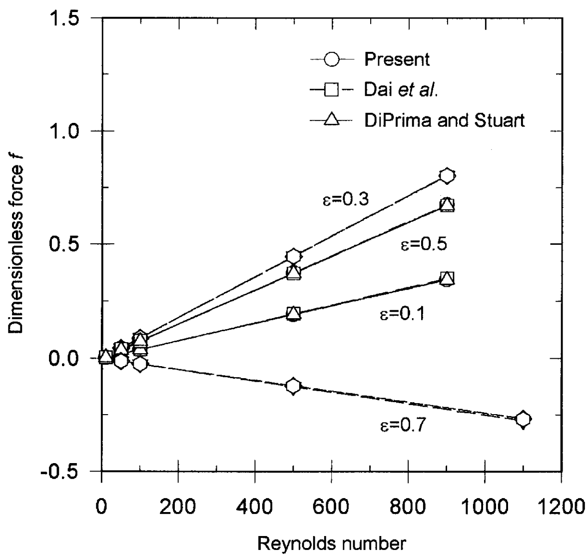

Figure 6. Comparison of numerical solutions at different eccentricity ratios and Reynolds numbers

3.2. *Viscous flow between eccentric rotating cylinders*

The flow between eccentric rotating cylinders is presented in this subsection. The two-dimensional steady state problem with constant viscosity is considered to compare with published results. Since our present numerical method is independent of the method utilized by Dai *et al.*,[15] it can be used to check the two solutions against one another. Numerical results for the problem are given in Figure 6, where the dimensionless force capacity f is given by

$$f = (f_x^2 + f_y^2)^{1/2}, \tag{23}$$

with $f_x = \int_0^\theta p \sin\theta \, d\theta$ and $f_y = -\int_0^\theta p \cos\theta \, d\theta$.

These results are compared with those of Dai *et al.*[15] and with the small-perturbation solutions of DiPrima and Stuart.[16] After examining Figure 6, we have concluded that the agreement among the three sets of data is very satisfactory. It is also noted that although DiPrima and Stuart assumed small eccentricity (ε) and small Reynolds number in their analysis, their results show unexpectedly good

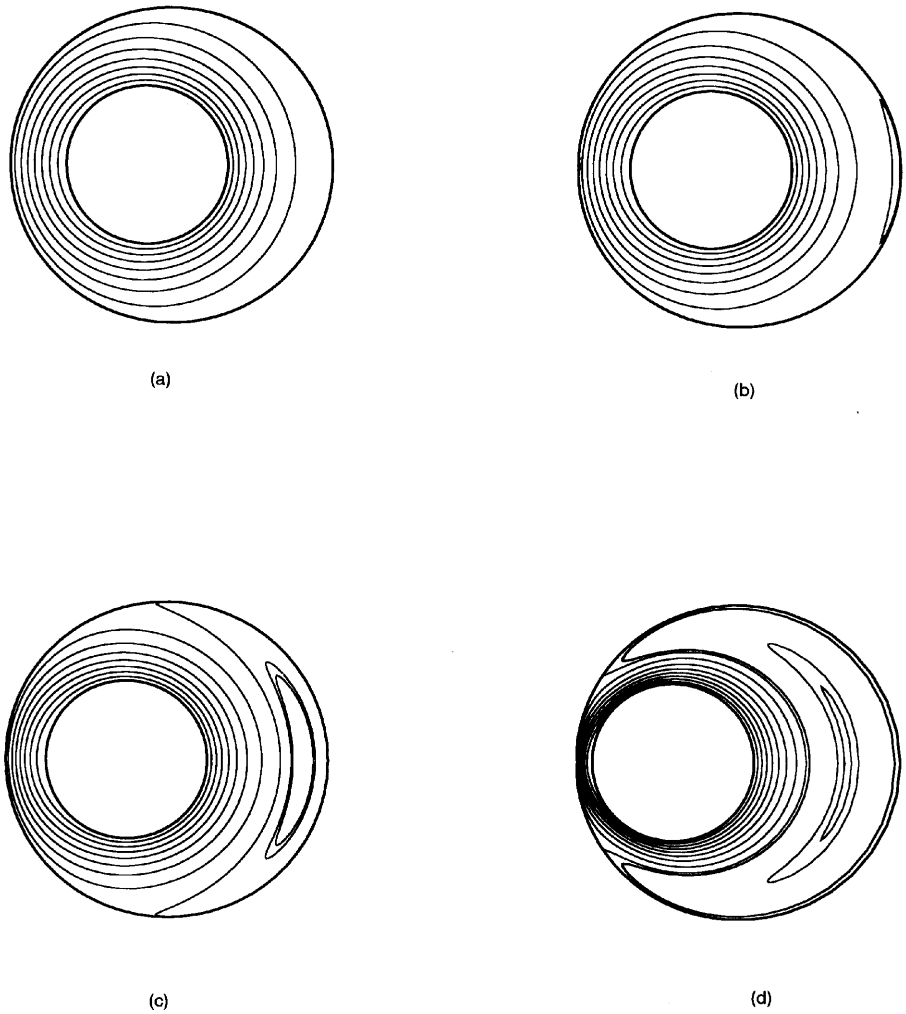

Figure 7. Streamline patterns for Stokes flow: (a) $\varepsilon = 0{\cdot}3$; (b) $\varepsilon = 0{\cdot}35$; (c) $\varepsilon = 0{\cdot}5$; (d) $\varepsilon = 0{\cdot}8$

agreement with the two sets of numerical results even at $\varepsilon = 0{\cdot}7$ over the whole range of laminar Reynolds numbers. Figure 7 shows the streamlines for a rotating inner and a stationary outer cylinder at various values of eccentricity. The results are the solution of creeping flow. The recirculation flow is symmetric relative to the lines of centres. The points of separation and reattachment show very good agreement with the conclusions of San Andres and Szeri.[17]

4. CONCLUSIONS

In this paper we presented the solution of the Navier–Stokes equations in primitive variable form by a mixed Galerkin formulation employing B-spline basis functions to compute two-dimensional flow. The formulation circumvents restrictions of the Babuska–Brezzi condition. In particular, the B-spline basis, which is superior to other splines in providing computer solutions to fluid flow problems, is accommodated. One of the advantages of the B-spline basis is that it has excellent approximation properties.

For one-dimensional problem the formulation that is called a mixed Galerkin formulation gives excellent agreement. For a two-dimensional problem it is noted that the effect of fluid inertia has been included for comparison of approximate analyses of two rotating cylinders, although inertia effects seem to be less important there. According to the tests, the solution of the procedure presented in this paper is reasonable without any mathematical manipulation of the pressure. We observed that in both cases the solution converges very fast.

REFERENCES

1. I. Babuska, 'Error bounds for finite element method', *Numer. Math.*, **16**, 322–333 (1971).
2. L. R. Abrahamsson, H. B. Keller and H. O. Kreiss, 'Difference approximations for singular perturbations of system of ordinary differential equations', *Numer. Math.*, **22**, 367 (1974).
3. I. Christie, D. F. Griffiths, A. R. Mitchell and O. C. Zienkiewicz, 'Finite element methods for second order differential equations with significant first derivatives', *Int. j. numer. meth. engng.*, **10**, 1389–1396 (1976).
4. J. C. Heinrich, P. S. Huyakorn, O. C. Zienkiewicz and A. R. Mitchell, 'An upwind finite element scheme for dimensional convective transport equation', *Int. j. numer. meth. engng.*, **11**, 131–143 (1977).
5. D. W. Keller, S. Nakazawa and O. C. Zienkiewicz, 'A note on upwinding and anisotropic balancing dissipation in finite element approximations to convective diffusion problems', *Int. j. numer. meth. engng.*, **15**, 1705–1711 (1980).
6. T. J. R. Hughes, L. P. Franca and M. Balestra, 'A new finite element formulation for computational fluid dynamics', *Comput. Meth. Appl. Mech. Engng.*, **59**, 85–99 (1986).
7. P. A. B. Sampaio, 'A Petrov–Galerkin formulation for the incompressible Navier–Stokes equations using equal order interpolation for velocity and pressure', *Int. j. numer. meth. engng.*, **31**, 1135–1149 (1991).
8. O. C. Zienkiewicz and J. Wu, 'Incompressibility without tears—how to avoid restrictions of mixed formulation', *Int. j. numer. meth. engng.*, **32**, 1189–1203 (1991).
9. R. L. Panton and H. B. Sallee, 'Spline function representations for computer solutions to fluid problems', *Comput. Fluids*, **3**, 257–269 (1975).
10. J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Non-linear Equations in Several Variables*, Academic, New York, 1970.
11. C. deBoor, *A Practical Guide to Splines*, Springer, Berlin, 1978.
12. L. E. Howle, 'Efficient implementation of a finite-difference/Galerkin method for simulation of large aspect ratio convection', *Numer. Heat Transfer B*, **26**, 105–114 (1994).
13. M. S. Krakow, 'Control volume finite element method for Navier–Stokes equations in vortex–streamfunction formulation', *Numer. Heat Transfer B*, **21**, 125–145 (1992).
14. H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 1968.
15. R. X. Dai, Q. Dong and A. Z. Szeri, 'Flow of variable viscosity fluid between eccentric rotating cylinders', *Int. J. Non-linear Mech.*, **27**, 367–389 (1992).
16. R. C. DiPrima and J. T. Stuart, 'Non-local effects in the stability of flow between eccentric rotating cylinders', *J. Fluid Mech.*, **54**, 393–415 (1972).
17. L. A. San Andres and A. Z. Szeri, 'Flow between eccentric, rotating cylinders', *J. Appl. Mech.*, **51**, 869–878 (1983).
18. I. Christie, K. R. Rajagopal and A. Z. Szeri, 'Flow of a non-Newtonian fluid between eccentric rotating cylinders', *Int. J. Engng. Sci.*, **25**, 1029–1047 (1987).

